DISCRETE MATHEMATICS I

B.MATH 2ND YEAR

SUPPLEMENTARY MID TERM EXAM

Instructions

- Part A contains 9 questions. Each question carries 5 marks each. Answer any 6.
- Part B contains 3 questions. Each question carries 10 marks each. Answer any 2.
- Time limit for the exam is 3 hours.
- You are allowed to name/quote and use any theorem, proposition, lemma or corollary proved in class.
- You are not allowed to quote and use problems discussed in class, assignments and quizzes without proof.

Notations

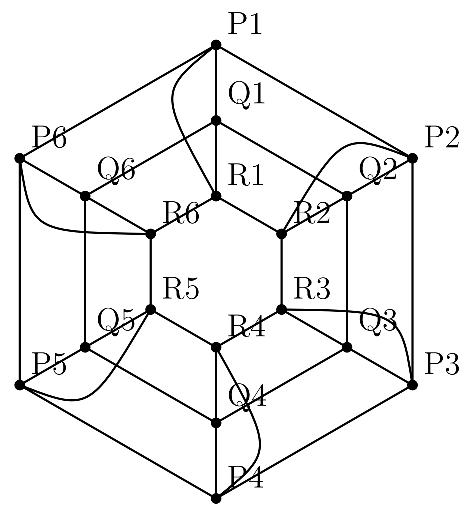
- $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- $[n] = \{1, 2, \ldots, n\}$, for $n \in \mathbb{N}$.
- Set of integers is $\mathbb{Z} = \mathbb{N} \cup \{-n \mid n \in \mathbb{N}\}$
- A positive integer is an element of the set $\mathbb{N} \setminus \{0\}$.
- Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then $\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{if } 0 \le k \le n \\ 0 & \text{otherwise.} \end{cases}$
- The sequence of Fibonacci numbers $\{F_n\}_{n \in \mathbb{N}}$ is defined by the recurrence relation
$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \ \forall \ n \in \mathbb{N}.$$
The ordinary generating function of $\{F_n\}_{n \in \mathbb{N}}$ is $\dfrac{x}{1 - x - x^2}$.
- A composition of n is a sequence $(a_1, a_2, \ldots, a_k)$ of positive integers such that $k \ge 0$ and $\sum_{i=1}^{k} a_i = n$. The numbers a_i are called parts of the composition. *Note that, if $k = 0$, then the sequence is an empty sequence, and in that case, $\sum_{i=1}^{k} a_i = 0$.*

Part A (5 marks per question, answer any 6)

1. Let M be an $2k \times 2k$ square matrix whose diagonal entries are all 0, and off diagonal entries are all either 1 or -1. Prove that M is invertible.

2. How many ordered triples (A, B, C) of subsets of an n-element set S satisfy $A \cap B \cap C = \varnothing$?

3. Consider the following simple undirected graph G :



(a) Does G contain a closed Eulerian trail?

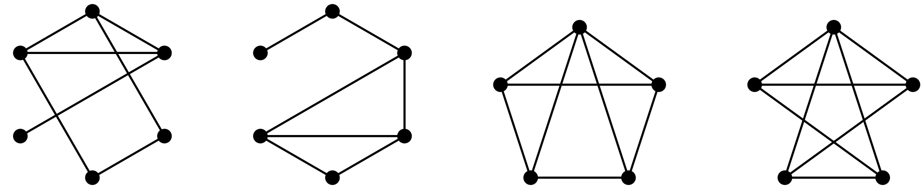
(b) Prove that G contains a Hamiltonian cycle.

4. 57 students took the Discrete Mathematics midterm examination. Every student copied from exactly 3 students each. Is it possible for every student to only have copied from students who copied from them?

5. Prove the following identity for every $n \in \mathbb{N}$:

$$\sum_{k \geq 0} \binom{n+k}{2k} 2^{n-k} = \frac{1}{3}\left(2^{2n+1} + 1\right)$$

6. Consider the following four simple undirected graphs :

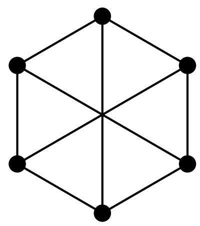
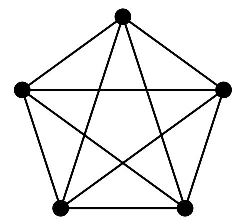


. G_1 G_2 G_3 G_4

(a) Is G_1 isomorphic to G_2? Justify your answer.

(b) Is G_3 isomorphic to G_4? Justify your answer.

7. Are the following two graphs bipartite? If not, find their chromatic numbers.

8. Prove that, a finite undirected graph $G = (V, E)$ is connected if and only if for every partition of V into two nonempty sets V_1 and V_2, there is an edge in E with one endpoint in V_1, and another endpoint in V_2.

9. Let S be a set of natural numbers of cardinality n. Prove that S contains a nonempty subset T such that the sum of elements of T is divisible by n.

Part B (10 marks per question, answer any 2)

10. (a) Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence defined by the recurrence relation

$$a_0 = 2, \qquad a_n = 2na_{n-1} - n! \;\forall\; n \geq 1.$$

Describe the exponential generating function of the sequence $\{a_n\}_{n\in\mathbb{N}}$.

(b) Let $\{c_n\}_{n\in\mathbb{N}}$ be a sequence defined by the recurrence relation

$$c_0 = 3, \quad c_1 = 4, \quad c_n = c_{n-1} + 6c_{n-2} \;\forall\; n \geq 2.$$

Find the value of c_n for all $n \in \mathbb{N}$.

11. (a) Let n be a positive integer. Prove that, given a piece of paper in the shape of an equilateral triangle, one can cut it into $8n+1$ smaller pieces which are all in the shape of equilateral triangles (Not necessarily of the same size).

(b) Let $m \geq 15$ be a positive integer. Prove that, given a piece of paper in the shape of an equilateral triangle, one can cut it into m smaller pieces which are all in the shape of equilateral triangles (Not necessarily of the same size).

12. (a) Let $n \in \mathbb{N} \setminus \{0\}$. Prove that the number of compositions of n with all parts > 1 is F_{n-1}.

(b) Let $n > 0$. Find the number of compositions of n with all parts even.